Total Market Equilibria

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Abstract

The total market containing all assets is in equilibrium where all investors have the same utility functions and hold the same fully diversified total market portfolio. This is not an equilibrium, however, where they have different utility functions, even if they are all risk averse. Then investors can all increase their utility by reallocating the market returns among themselves on a non pro-rata basis. Even in a perfect market the utility maximizing investment strategy for risk averse investors with different utility functions requires them to bear idiosyncratic risk, providing a role for asset transformation. The maximum or minimum asset prices at which an investor will transact in pursuance of greater portfolio utility are unique to that investor and the existing market state.
1 Introduction

The capital asset pricing model (CAPM) adopts the classic set of assumptions; that markets are perfect, and that investors are risk averse, have a common one period investment horizon, and maximize a utility function on asset return. It also, however, makes specific restrictive assumptions; investors maximize utility on the first two moments of the return distribution only, and there is a risk free asset exogeneous to the market. The CAPM’s major result is that this market will be in equilibrium where all investors hold the fully diversified market portfolio. No investor could be induced to trade away from the market portfolio.

Despite its specific assumptions the CAPM result has almost evolved the status of a general law; if markets are perfect then risk averse investors will maximize their utility only if they are fully diversified in risky assets, bearing only systematic risk. In fact the standard technique for identifying market imperfections is to identify investment strategies that persistently generate risky asset returns in excess of the returns of the fully diversified risky market portfolio.

This paper, however, assesses the robustness of this result even under the classic set of assumptions, by relaxing only the CAPM specific assumptions. Investors are assumed to maximize expected utility on the full distribution of asset return and the market is expanded to a total market which includes all assets, irrespective of the form of their return distributions. In this total market a risk free asset is one with a contractual payoff less than the minimum value in the distribution of the market payoffs against which it has the priority claim, and so all risk free assets are endogenous.

An investor holds the fully diversified total market portfolio (TMP) when it holds each asset in the same proportion that it contributes to total market investment. The test of the generality of the CAPM result is whether risk averse investors, who simultaneously hold the TMP, could increase their utility by reallocating its returns amongst themselves on a non pro-rata basis, such that the returns on their resulting portfolios are imperfectly correlated. If so, then full diversification would not maximize the utility of risk averse investors. Optimal investing, even in a perfect market, would require them to bear idiosyncratic risk.

One important assumption is made, that an investor applies the same utility function to the return distributions of all assets. Given this, two scenarios are considered. First, investors all have the same utility function, and, second, investors have different utility functions.

In the highly restrictive first scenario it is found that the TMP is a market equilibrium portfolio. In the more general second scenario, however, it is found that the TMP is not a market equilibrium portfolio. In fact it is found that the higher utility set of portfolios to which the market will evolve from the TMP will have return profiles in the same general form as those resulting from the writing of a covered option on the TMP.
2 Discrete probability model of return distributions

The two major results of this paper are derived from the following simple model of asset return distributions, called the discrete probability model.

Certainty, or probability 1, is represented by a set of $\rho$ discrete and fundamental probability units each with probability mass $\frac{1}{\rho} \implies 0$ and denoted $\omega_1, \omega_2, \ldots, \omega_\rho$. The probability units might be conceptualized as representing the set of discrete states of the world existing at the investment horizon.

The distribution $y$ comprises a set of locations which represent the return of the asset. Each probability unit $\omega_z$ maps to a return location which is denoted $R(\omega_z, y)$ and so the probability of return $r$ in distribution $y$ is given by $\sum_{z=1}^{\rho} \text{if}(r = R(\omega_z, y), \frac{1}{\rho}, 0)$.

The results of this paper are are generated as follows.

3 Identical investor utility functions

Consider an investor whose preferences for return distributions are risk averse. In terms of the discrete probability model, this investor’s utility function $U(.)$ satisfies the classic inequality

$$U \left( \sum_{z=1}^{\rho} \frac{1}{\rho} R(\omega_z, y) \right) > \sum_{z=1}^{\rho} \frac{1}{\rho} U(R(\omega_z, y))$$  \hspace{1cm} (1)

where $y$ is any asset with random returns (a risky asset). That is, the investor is risk averse if the utility of the expected return of asset $y$ is greater than the expected utility of that asset’s returns.

The elements of this inequality can, however, be redefined to create a new Equation (2)

$$U \left( \sum_{y=1}^{n} W(y).R(\omega_z, y) \right) > \sum_{y=1}^{n} W(y).U(R(\omega_z, y))$$  \hspace{1cm} (2)

where a portfolio comprises $n$ assets $y$, each with portfolio weighting $W(y)$ such that $\sum_{y=1}^{n} W(y) = 1$, and $R(\omega_z, a) \neq R(\omega_z, b)$. That is, in a state of the world represented by probability unit $\omega_z$, the utility of the return of a portfolio of assets is greater than the weighted average utility of the return on each of the portfolio assets.

This sums across the entire set of $\rho$ probability units as

$$\sum_{z=1}^{\rho} \frac{1}{\rho} \left( U \left( \sum_{y=1}^{n} W(y).R(\omega_z, y) \right) \right) > \sum_{z=1}^{\rho} \frac{1}{\rho} \left( \sum_{y=1}^{n} W(y).U(R(\omega_z, y)) \right)$$  \hspace{1cm} (3)

which can be restated as
Equation (4) gives the expression of investor risk aversion as a portfolio effect, which is called the general portfolio effect (GPE). The expected utility of a portfolio of imperfectly correlated assets is greater than the weighted average expected utility of those assets.

The GPE holds even where one portfolio asset is risk free, with distribution denoted $R_f$, such that $R(z, R_f) = R(z, R_f) = \ldots = R(z, R_f) = RF$. Let $R_f$ and a risky asset $y$ be combined in a portfolio with weightings $W(R_f) + W(y) = 1$. Substituting into Equation (4) gives

$$\sum_{z=1}^{\rho} \frac{1}{\rho} \left( U \left( \sum_{y=1}^{n} W(y) R(z, y) \right) \right) > \sum_{y=1}^{n} W(y) \left( \sum_{z=1}^{\rho} \frac{1}{\rho} U(R(z, y)) \right)$$

(4)

Now consider two risk averse investors with identical utility functions. The investors are both invested in the TMP and so have identical expected utility. They might, however, reallocate the returns of the TMP between themselves on a non pro-rata basis to create new portfolios with imperfectly correlated return distributions, respectively $A_1$ and $A_2$, such that

$$R(z, A_1) \neq R(z, A_2)$$

and

$$R(z, TMP) = W(A_1).R(z, A_1) + W(A_2).R(z, A_2)$$

(6)

Substituting portfolio assets $A_1$ and $A_2$ into Equation (4) gives

$$\sum_{z=1}^{\rho} \frac{1}{\rho} \left( U \left( W(A_1) R(z, A_1) + W(A_2) R(z, A_2) \right) \right) > W(A_1) \left( \sum_{z=1}^{\rho} \frac{1}{\rho} U(R(z, A_1)) \right) + W(A_2) \left( \sum_{z=1}^{\rho} \frac{1}{\rho} U(R(z, A_2)) \right)$$

which, from Equation (6), gives

$$\sum_{z=1}^{\rho} \frac{1}{\rho} U(R(z, A))$$

(7)

$$> W(A_1) \left( \sum_{z=1}^{\rho} \frac{1}{\rho} U(R(z, A_1)) \right) + W(A_2) \left( \sum_{z=1}^{\rho} \frac{1}{\rho} U(R(z, A_2)) \right)$$

Equation (7) is interpreted as follows. The expected utility of the TMP is greater than the weighted average expected utility of the resulting imperfectly correlated portfolios $A_1$ and $A_2$ into which it is decomposed. From Equation (5), this result holds even where one of the resulting portfolios is risk free.
It follows that if the returns of the TMP were to be allocated among investors with identical utility functions on a non pro-rata basis then at least one investor must have a resulting expected utility less than it had from the TMP. It cannot be induced to trade away from the TMP. This market is, therefore, in equilibrium where all investors hold the TMP.

4 Non-identical investor utility functions

Consider two risk averse investors with non-identical utility functions $U_1(.)$ and $U_2(.)$ and equal initial investments. Let the slope of the tangents to their utility functions at any return location $r$ be denoted $T_1\{r\}$ and $T_2\{r\}$ respectively.

Assume that investor 1 is more risk averse than investor 2 such that

$$\frac{T_1\{r\}}{T_2\{r\}} > \frac{T_1\{r + \alpha\}}{T_2\{r + \alpha\}} > 1 \quad \text{where } \alpha > 0 \quad (8)$$

Start with both investors holding the TMP. Now take any two probability units $\omega_z$ and $\omega_y$ with locations on the TMP return distribution

$$R(\omega_z, TMP)$$

and

$$R(\omega_y, TMP)$$

where

$$R(\omega_z, TMP) < R(\omega_y, TMP)$$

4.1 Probability unit $\omega_z$

Reallocate the lower TMP return $R(\omega_z, TMP)$ among the investors such that they have resulting return locations for $\omega_z$

Investor 1 $R(\omega_z, A1) = R(\omega_z, TMP) + x$

and

Investor 2 $R(\omega_z, A2) = R(\omega_z, TMP) - x$

Let the slope of any chord to their utility functions be denoted

Investor 1 $C_1\{r, r + c\}$

and

Investor 2 $C_2\{r, r + c\}$

where the chord intersects the utility functions at return locations $r$ and $r + c$.

The change in each investor’s utility from the reallocation of the return $\omega_z$ is therefore given by

Investor 1 $C_1\{R(\omega_z, TMP), R(\omega_z, A1)\}.x$
and
\[ \text{Investor 2 } - C_2 \{ R(\omega_z, TMP), R(\omega_z, A2) \}.x \]

Now assume that \( x \) approaches zero. The chord slopes can now be approximated by the tangents
\[ \text{Investor 1 } C_1 \{ R(\omega_z, TMP), R(\omega_z, A1) \} \approx T_1 \{ R(\omega_z, TMP) \} \]
and
\[ \text{Investor 2 } - C_2 \{ R(\omega_z, TMP), R(\omega_z, A2) \} \approx -T_2 \{ R(\omega_z, TMP) \} \]

The change in each investor’s utility from the reallocation of the return \( \omega_z \) is therefore approximated by
\[ \text{Investor 1 } T_1 \{ R(\omega_z, TMP) \}.x \quad (9) \]
and
\[ \text{Investor 2 } - T_2 \{ R(\omega_z, TMP) \}.x \quad (10) \]

### 4.2 Probability unit \( \omega_y \)

Next reallocate the higher TMP return \( R(\omega_y, TMP) \) among the investors such that they have resulting return locations for \( \omega_y \) of
\[ \text{Investor 1 } R(\omega_y, A1) = R(\omega_y, TMP) - w \]
and
\[ \text{Investor 2 } R(\omega_y, A2) = R(\omega_y, TMP) + w \]

Let
\[ w = x \frac{T_1 \{ R(\omega_z, TMP) \}}{T_1 \{ R(\omega_y, TMP) \}} \quad (11) \]

The change in each investor’s utility from the reallocation of the return \( \omega_y \) is approximated by
\[ \text{Investor 1 } - T_1 \{ R(\omega_y, TMP) \}.w \quad (12) \]
and
\[ \text{Investor 2 } T_2 \{ R(\omega_y, TMP) \}.w \quad (13) \]

### 4.3 Probability units \( \omega_z \) and \( \omega_y \), small \( x \) and \( w \)

For investor 1, the net change in expected utility across the pair of probability units approaches, from Equations (9), (11) and (12),
\[ T_1 \{ R(\omega_z, TMP) \}.x - T_1 \{ R(\omega_y, TMP) \}.w \]
\[ = T_1 \{ R(\omega_z, TMP) \}.x - T_1 \{ R(\omega_y, TMP) \}.w \frac{T_1 \{ R(\omega_z, TMP) \}}{T_1 \{ R(\omega_y, TMP) \}} \]
\[ = T_1 \{ R(\omega_z, TMP) \}.x - T_1 \{ R(\omega_z, TMP) \}.x \frac{T_1 \{ R(\omega_z, TMP) \}}{T_1 \{ R(\omega_y, TMP) \}} \]
\[ = T_1 \{ R(\omega_z, TMP) \}.x - T_1 \{ R(\omega_z, TMP) \}.x \]
\[ = 0 \]
For investor 2 it approaches, from Equations (10), (11) and (13),

\[ T_2\{R(\omega_y, TMP)\}.w - T_2\{R(\omega_z, TMP)\}.x \]

\[ = T_2\{R(\omega_y, TMP)\}.x \cdot \frac{T_1\{R(\omega_z, TMP)\}}{T_1\{R(\omega_y, TMP)\}} - T_2\{R(\omega_y, TMP)\}.x \]

\[ = x\cdot\left[\frac{T_1\{R(\omega_z, TMP)\}}{T_1\{R(\omega_y, TMP)\}}\right] T_2\{R(\omega_y, TMP)\} - T_2\{R(\omega_z, TMP)\} \]

Now recall, from Equation (8), that

\[ \frac{T_1\{R(\omega_z, TMP)\}}{T_1\{R(\omega_y, TMP)\}} > \frac{T_2\{R(\omega_z, TMP)\}}{T_2\{R(\omega_y, TMP)\}} \]

It follows that

\[ \frac{T_1\{R(\omega_z, TMP)\}}{T_1\{R(\omega_y, TMP)\}} - T_2\{R(\omega_z, TMP)\} - T_2\{R(\omega_y, TMP)\} \]

\[ > 0 \]

As \( x \) is also greater than 0 then Equation (15) > 0 and so investor 2’s net change in expected utility across the pair of probability units is greater than zero.

Recall that this result is found where

\[ w = x \cdot \frac{T_1\{R(\omega_z, TMP)\}}{T_1\{R(\omega_y, TMP)\}} \]

If some of investor 2’s net increase in expected utility is shared with investor 1 by reducing \( w \) by a small amount so that

\[ w \Rightarrow x \cdot \frac{T_1\{R(\omega_z, TMP)\}}{T_1\{R(\omega_y, TMP)\}} \]

then the net change in expected utility across the pair of probability units for both investor 1 and investor 2 will be positive.

### 4.4 Probability units \( \omega_z \) and \( \omega_y \), larger \( x \) and \( w \)

Now calibrate \( w \) such that, for any large value \( x \), the net change in expected utility of investor 1 across the two probability units is zero. Denote this value \( \hat{w} \). That is

Investor 1 \[ C_1\{R(\omega_z, TMP), R(\omega_z, A1)\}.x - C_1\{R(\omega_y, TMP), R(\omega_y, A1)\}.\hat{w} = 0 \]

This can be restated to give
As $x$ increases, the slope of the tangents less approximate the slope of the respective chords. That is

$$T_1 \{R(w, TMP)\} > C_1 \{R(w, TMP), R(w, A1)\}$$

$$T_2 \{R(w, TMP)\} < C_2 \{R(w, TMP, R(w, A2)\}$$

$$T_1 \{R(w, TMP)\} < C_1 \{R(w, TMP), R(w, A1)\}$$

$$T_2 \{R(w, TMP)\} > C_2 \{R(w, TMP), R(w, A2)\}$$

If $x$ becomes sufficiently large then, denoting this value $\hat{x}$,

$$\frac{C_1 \{R(w, TMP, R(w, A1)\}}{C_2 \{R(w, TMP), R(w, A2)\}} = \frac{C_1 \{R(w, TMP), R(w, A1)\}}{C_2 \{R(w, TMP), R(w, A2)\}}$$

The net change in expected utility for investor 2 is given by

Investor 2 $\quad C_2 \{R(w, TMP), R(w, A2)\}.\hat{w} - C_2 \{R(w, TMP), R(w, A2)\}.\hat{x}$

Substituting Equation (16) into Equation (17) gives

$$\frac{C_1 \{R(w, TMP), R(w, A1)\}}{C_2 \{R(w, TMP), R(w, A2)\}} = \frac{C_1 \{R(w, TMP), R(w, A1)\}}{C_2 \{R(w, TMP), R(w, A2)\}}$$

$$\frac{C_1 \{R(w, TMP), R(w, A1)\}.\hat{w}}{\hat{x}} = \frac{C_1 \{R(w, TMP), R(w, A1)\}.\hat{w}}{\hat{x}} = \frac{C_2 \{R(w, TMP), R(w, A2)\}}{C_2 \{R(w, TMP), R(w, A2)\}}$$
\[ C_2\{R(z, TMP), R(z, A2)\} \cdot \bar{x} = C_2\{R(y, TMP), R(y, A2)\} \cdot \bar{w} \]

and so, from Equation (18),

Investor 2 \[ C_2\{R(y, TMP), R(y, A2)\} \cdot \bar{w} - C_2\{R(z, TMP), R(z, A2)\} \cdot \bar{x} = 0 \]

The net change in investor 2’s expected utility across the two probability units is zero.

4.5 Result

Recall that, for the most risk averse investor, its lower TMP return is increased by \(x\) and its higher TMP return is reduced by \(w\) in the reallocation of returns. For the least risk averse investor, its lower TMP return is decreased by \(x\) and its higher TMP return is increased by \(w\) in the reallocation of returns.

Where \(x \rightarrow 0\) then it is possible for the expected utility of both investors to be greater compared with their expected utilities under the TMP. It is only at some positive value \(\bar{x}\) that their weighted average expected utilities must be unchanged and above \(\bar{x}\) that their weighted average expected utilities must be less compared with their weighted average expected utilities under the TMP.

There thus exists a range of values for \(x < \bar{x}\) at which both the investors have incentive to reallocate the returns of the TMP among themselves on a non pro-rata basis. Where \(x < \bar{x}\) then, for both investors to increase their expected utility, \(w\) must be greater than \(x\), and so we can also make the following observations about the resulting expected utility maximizing portfolios.

The new portfolio of the most risk averse investor has, relative to the TMP, lower variance of return and lower expected return, with a distribution of return more skewed in the direction of lower return. The new portfolio of the least risk averse investor has, relative to the TMP, greater variance of return and greater expected return, with a distribution of return skewed more in the direction of higher return.

It can be observed that this is the same change in the relative form of investor returns as would result from the writing of a covered option on an asset by the most risk averse investor.

The significant result is that the returns of the investors’ resulting expected utility maximizing portfolios are imperfectly correlated. They are not fully diversified and the investors are therefore bearing idiosyncratic risk.


5 Conclusion

This paper shows that, even in perfect markets, risk averse investors do not maximize utility when fully diversified except when severe restrictive assumptions are imposed; either that they all have the same utility function, or the CAPM assumptions that they maximize utility only on the first two moments of the return distribution and that there is an exogenous risk free asset. If these assumptions are relaxed then investors can be induced to trade away from the fully diversified market portfolio in order to mutually maximize their utilities.

The work in this paper has a number of fundamental implications, which might be distilled into three, as follows.

First; there is a portfolio effect, the general portfolio effect, when uncorrelated assets are combined in a portfolio. This effect is general in that it is expressed directly in terms of investor utility and it describes the result of combining assets with different forms of return distribution, including risky and risk free assets. While it shows there is a benefit from diversification, this benefit is mitigated by any difference between the stand-alone expected utilities of the assets, which skews the weighting of assets in any investor’s optimal utility maximizing portfolio towards those with highest stand-alone expected utility for the investor.

Second; asset transformation, major examples of which include the leveraging of firms and the writing of options on market assets, has a role in maximizing investor utility, notwithstanding that the returns of the resulting assets are uncorrelated and so the investors who hold them bear idiosyncratic risk.

Third; for given available market returns, an investor can always have higher expected utility where the market contains other investors with utility functions different from its own.

Ultimately this work confirms that there can be no short cuts in the process of asset pricing, even in a perfect market. The rate at which an investor will be willing to trade one asset for another, or the allocation of returns of a transformed asset it would be willing to accept, will be that at which the expected utility of return of its resulting portfolio will be greater than the expected utility of its existing portfolio. The pricing at which it will transact will thus be unique to the investor’s own utility function and the distribution of returns on both its existing portfolio and the resulting portfolio.

This suggests significant scope for further work, but two areas stand out in particular. First, testing the reasonableness of the assumption that investors apply the same utility function to all assets. And, second, determining whether asset markets with given returns and investors with given utility functions will evolve from different starting states to the same equilibria, or only to unique local equilibria.
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